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Unified solutions of heat diffusion in a finite region involving a surface film of finite heat capacity

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Abstract—The finite integral transform technique is further developed and adopted to obtain an exact solution for transient heat diffusion in an arbitrary finite region having a surface film of finite heat capacity. A splitting-up procedure that accelerates the convergence of the series solutions developed here is described. Solutions for a vast number of specific situations of practical interest can readily be obtained as special cases from the general results presented in this work.

INTRODUCTION

THE INTEGRAL transform technique provides a systematic and powerful approach for solving partial differential equations encountered in heat and mass diffusion. In a recent monograph [1], this technique has been extensively used to develop general solutions for seven different classes of heat and mass diffusion problems. The CLASS I problem is related to unsteady heat or mass diffusion in a finite region of arbitrary geometry subjected to generalized boundary and initial conditions. As a further generalization of this class of problem, we consider here a finite region of arbitrary geometry surrounded by a thin layer of high conductivity material which is subjected to convection with an external ambient. In such situations this outer layer can be lumped in the direction normal to the surface of the inner region. It is assumed that the lateral conduction is negligible in the lumped layer. Then the heat conduction equation for the outer layer becomes a more general boundary condition for the inner region and includes a time derivative term resulting from the lumping.

Carslaw and Jaeger [2], Crank [3] and Mikhailov [4] discuss some practical applications of this type of boundary conditions. In a recent work Beck [5] used Green's function approach to analyze heat diffusion problems of this type.

In this work we consider a sufficiently general heat diffusion problem which includes as special cases all the problems belonging to this class that are studied in the literature.

STATEMENT OF THE PROBLEM

Consider the following boundary value problem of a finite region

$$w(\underline{x})\frac{\partial T(\underline{x},t)}{\partial t} + LT(\underline{x},t) = P(\underline{x},t), \quad \underline{x}\in V \qquad (1a)$$

subject to the boundary condition

$$\gamma(\underline{x})\frac{\partial T(\underline{x},t)}{\partial t} + BT(\underline{x},t) = \phi(\underline{x},t), \quad \underline{x}\in S \qquad (1b)$$

and the initial conditions

$$T(\underline{x},t) = f(\underline{x}), \quad \underline{x}\in V \qquad (1c)$$

$$T(\underline{x},t) = f_s(\underline{x}), \quad \underline{x}\in S \qquad (1d)$$

where the linear operators L and B are defined as

$$L \equiv -\nabla\cdot[k(\underline{x})\nabla]+d(\underline{x}) \qquad (2a)$$

$$B \equiv \alpha(\underline{x})+\beta(\underline{x})k(\underline{x})\frac{\partial}{\partial \underline{n}}. \qquad (2b)$$

The first term in the boundary condition (1b) includes the possibility of a high conductivity surface film, but heat flow parallel to the surface inside the film is neglected. The problem (1) for $\gamma(\underline{x}) = 0$ becomes the CLASS I problem of ref. [1].

The appropriate eigenvalue problem for the solution of the system (1) is taken as

$$\mu^2 w(\underline{x})\psi(\underline{x}) = L\psi(\underline{x}), \quad \underline{x}\in V \qquad (3a)$$

$$\mu^2 \gamma(\underline{x})\psi(\underline{x}) = B\psi(\underline{x}), \quad \underline{x}\in S \qquad (3b)$$

where the linear operators L and B are defined by equations (2).

THE INTEGRAL TRANSFORM PAIR

The problem (3) does not belong to the conventional Sturm–Liouville system. Therefore, the first step in the analysis is to establish the orthogonality condition appropriate to the system (3) as now described. Let $\psi_i(\underline{x})$ and $\psi_j(\underline{x})$ be two different eigenfunctions corresponding to two different eigenvalues μ_i and μ_j. Equation (3a) is written for $\psi_i(\underline{x})$ *and* $\psi_j(\underline{x})$ and multiplied respectively by $\psi_j(\underline{x})$ and $\psi_i(\underline{x})$. Then the results are subtracted and integrated over the volume. Equation (3b) is written for $\psi_i(\underline{x})$ and $\psi_j(\underline{x})$ and multiplied respectively by $\psi_j(\underline{x})/\beta(\underline{x})$ and $\psi_i(\underline{x})/\beta(\underline{x})$. Both sides of the results obtained above are added to obtain the following

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NOMENCLATURE

B	boundary condition operator defined by equation (2b)
$d(\underline{x}), d(x)$	prescribed functions
d_p, d_ϕ	prescribed coefficients
$\frac{\partial}{\partial \underline{n}}$	derivative in the direction of outward drawn normal to the surface S
$f(\underline{x}), f_s(\underline{x})$	initial temperature distribution in V and S, respectively
$f(x), f_k$	one-dimensional initial temperature distribution in $x_0 < x < x_1$ and at $x = x_k$ $(k = 0, 1)$, respectively
$\bar{f}_i$	integral transform of the initial condition defined by equation (9)
$g_i(t)$	the function defined by equation (8b)
I_ϕ, I_p, I_j, I_f	constants defined by equations (14c), (15c), (16d) and (17d)
$k(\underline{x}), k(x)$	prescribed functions
L	linear operator defined by equation (2a)
N_i	normalization integral defined by equation (5)
$P(\underline{x}, t), P(x, t)$	prescribed volumetric source functions
$P_e(\underline{x}), P_j(\underline{x})$	functions introduced in equation (12a)
$P_e(x), P_j(x)$	functions introduced in equation (22a)
S	boundary surface of V
$T(\underline{x}, t), T(x, t)$	temperatures
$T_{av}(t)$	average temperature defined by equations (11) and (20e) for the multi- and one-dimensional regions, respectively
$\bar{T}_i$	integral transform of temperature defined by equation (7a)
$T_\phi(\underline{x}), T_p(\underline{x}), T_j(\underline{x}), T_t(\underline{x}, t)$	temperature functions defined by equations (14), (15), (16) and (17), respectively
$T_\phi(x), T_p(x), T_j(x), T_t(x, t)$	temperature functions defined by equations (24), (25), (26) and (27), respectively
V	arbitrary finite region
$w(\underline{x}), w(x)$	prescribed functions
$\underline{x}, x$	space coordinate in multi- and one-dimension, respectively.

Greek letters

$\alpha(\underline{x}), \beta(\underline{x}), \gamma(\underline{x})$	boundary condition coefficients introduced in equations (2b) and (1b)
$\alpha_k, \beta_k, \gamma_k$	boundary coefficients introduced in equations (19b, c)
δ_{ij}	Kronecker delta
μ_i	eigenvalue
$\phi(\underline{x}, t), \phi_k(t)$	source function on the boundary surface S and at $x = x_k$ $(k = 0, 1)$, respectively
$\phi_e(\underline{x}), \phi_j(\underline{x})$	functions introduced in equation (12b)
$\phi_e(x_k), \phi_j(x_k)$	functions introduced in equations (22b, c)
$\psi_i(\underline{x}), \psi_i(x)$	eigenfunctions.

orthogonality condition

$$\int_V w(\underline{x})\psi_i(\underline{x})\psi_j(\underline{x})\,dv + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\psi_i(\underline{x})\psi_j(\underline{x})\,ds = \delta_{ij}N_i \quad (4)$$

where the normalization integral is given by

$$N_i = \int_V w(\underline{x})\psi_i^2(\underline{x})\,dv + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\psi_i^2(\underline{x})\,dx. \quad (5)$$

We note that, for $\gamma(\underline{x}) = 0$, the second term of the LHS of equation (4) vanishes and the new orthogonality condition presented here coincides with the well known Sturm–Liouville orthogonality condition.

We now consider the representation of the functions $T(\underline{x}, t)$ in the forms

$$T(\underline{x}, t) = \sum_{i=1}^{\infty} A_i(t)\psi_i(\underline{x}), \quad \underline{x} \in V \quad \text{or} \quad \underline{x} \in S. \quad (6)$$

Both sides of equation (6) are multiplied by $w(\underline{x})\psi_j(\underline{x})$ and integrated over the volume; both sides of equation (6) are multiplied by $\psi_j(\underline{x})\gamma(\underline{x})/\beta(\underline{x})$ and integrated over the surface S, and the results are added. In view of the orthogonality relation (4), the summation on the RHS of equation (6) vanishes and A_i is determined. We introduce the expression thus obtained for A_i into equation (6) and the resulting expansion for $T(\underline{x}, t)$ is split up into two parts to define the integral transform pair as:

Integral transform

$$\bar{T}_i(t) = \int_V w(\underline{x})\psi_i(\underline{x})T(\underline{x}, t)\,dv + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\psi_i(\underline{x})T(\underline{x}, t)\,ds; \quad (7a)$$

Inversion formula

$$T(\underline{x}, t) = \sum_{i=1}^{\infty} \frac{\psi_i(\underline{x})}{N_i}\bar{T}_i(t). \quad (7b)$$

METHOD OF SOLUTION

The integral transform pair defined by equations (7) is now applied to solve the general problem (1). Both sides of equations (1a) and (3a) are multiplied, respectively, by $\psi_i(\underline{x})$ and $T(\underline{x}, t)$, and integrated over the volume; then both sides of equations (1b) and (3b) are multiplied, respectively, by $\psi_i(\underline{x})/\beta(\underline{x})$ and $T(\underline{x}, t)/\beta(\underline{x})$ and integrated over the surface S; and all of these four results are added. Finally we obtain the following ordinary differential equation for the transform of the temperature $\tilde{T}_i(t)$

$$\frac{\mathrm{d}\tilde{T}_i(t)}{\mathrm{d}t} + \mu_i^2 \tilde{T}_i(t) = g_i(t) \tag{8a}$$

where

$$g_i(t) = \int_V \psi_i(\underline{x}) P(\underline{x}, t)\,\mathrm{d}v + \int_S \psi_i(\underline{x}) \frac{\phi(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s. \tag{8b}$$

The initial condition for this equation is determined by constructing the transform of the initial conditions (1c) and (1d) according to the transform (7a). We find

$$\tilde{T}_i(0) \equiv \tilde{f}_i = \int_V w(\underline{x})\psi_i(\underline{x}) f(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} \psi_i(\underline{x}) f_s(\underline{x})\,\mathrm{d}v. \tag{9}$$

Equation (8) is solved subject to the initial condition (9) and the result is introduced into the inversion formula (7b). Then the solution of the problem (1) becomes

$$T(\underline{x}, t) = \sum_{i=1}^{\infty} \frac{\psi_i(\underline{x})}{N_i} \mathrm{e}^{-\mu_i^2 t} \left\{ \tilde{f}_i + \int_0^t \mathrm{e}^{\mu_i^2 t'} g_i(t')\,\mathrm{d}t' \right\} \tag{10}$$

where the normalization integral N_i, the function $g_i(t)$ and the transform of the initial condition $\tilde{f}_i$ are defined, respectively, by equations (5), (8b) and (9). The $\psi_i(\underline{x})$ and μ_i are the eigenfunction and eigenvalues of the eigenvalue problem (3).

If $d(\underline{x}) = 0$ and $\alpha(\underline{x}) = 0$, then $\mu_0 = 0$ is also an eigenvalue with the corresponding eigenfunction ψ_0 = constant. For this special case, the solution (10) includes an additional term corresponding to $\mu_0 = 0$ and ψ_0 = constant, namely

$$T_{\mathrm{av}}(t) = \left\{ \int_V w(\underline{x}) f(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} f_s(\underline{x})\,\mathrm{d}s + \int_0^t \left[\int_v P(\underline{x}, t')\,\mathrm{d}v + \int_S \frac{\phi(\underline{x}, t')}{\beta(\underline{x})}\,\mathrm{d}s \right] \mathrm{d}t' \right\} \Big/ \left[\int_V w(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s \right]. \tag{11a}$$

This result for $T_{\mathrm{av}}(t)$, valid only for the case $d(\underline{x}) = \alpha(\underline{x}) = 0$, coincides with the mean potential defined in general as

$$T_{\mathrm{av}}(t) \equiv \frac{\int_V w(\underline{x}) T(\underline{x}, t)\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} T(\underline{x}, t)\,\mathrm{d}s}{\int_V w(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}x}. \tag{11b}$$

SPLITTING UP THE GENERAL PROBLEM

Although the general solution given above is exact, it is desirable, for computational purposes, to devise a scheme that permits the splitting up of the above problem into simpler ones, in order to replace the slowly convergent part of the solution with closed form expressions. A convenient scheme is to assume that the energy generation term $P(\underline{x}, t)$ and the non-homogeneous part of the boundary conditions $\phi(\underline{x}, t)$ are represented by exponentials and q-order polynomials of time as given by

$$P(\underline{x}, t) = P_e(\underline{x})\,\mathrm{e}^{-d_p t} + \sum_{j=0}^{q} P_j(\underline{x}) t^j, \quad \underline{x} \in V \tag{12a}$$

$$\phi(\underline{x}, t) = \phi_e(\underline{x})\,\mathrm{e}^{-d_\phi t} + \sum_{j=0}^{q} \phi_j(\underline{x}) t^j, \quad \underline{x} \in S \tag{12b}$$

where d_p and d_ϕ are constants. Then the solution $T(\underline{x}, t)$ of the general problem (1) can be split up into the solution of simpler problem as

$$T(\underline{x}, t) = T_\phi(\underline{x})\,\mathrm{e}^{-d_\phi t} + T_p(\underline{x})\,\mathrm{e}^{-d_p t} + \sum_{j=0}^{q} T_j(\underline{x}) t^j + T_t(\underline{x}, t), \quad \underline{x} \in V. \tag{13}$$

If $d(\underline{x}) = \alpha(\underline{x}) = 0$ then in the RHS of equation (13) appears the additional term defined by equation (11a).

The function $T_\phi(\underline{x})$ is the solution of the following steady-state problem

$$w(\underline{x}) I_\phi + L T_\phi(\underline{x}) = d_\phi w(\underline{x}) T_\phi(\underline{x}), \quad \underline{x} \in V \tag{14a}$$

$$\gamma(\underline{x}) I_\phi + B T_\phi(\underline{x}) = d_\phi \gamma(\underline{x}) T_\phi(\underline{x}) + \phi_e(\underline{x}), \quad \underline{x} \in S \tag{14b}$$

In general $I_\phi = 0$, except for the case $d(\underline{x}) = \alpha(\underline{x}) = 0$ when it is given by

$$I_\phi = \frac{\int_S \frac{\phi_e(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s}{\int_V w(\underline{x}) + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s} \tag{14c}$$

and the following additional condition appears

$$\int_V w(\underline{x}) T_\phi(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} T_\phi(\underline{x})\,\mathrm{d}x = 0. \tag{14d}$$

The function $T_p(\underline{x})$ is the solution of the following steady-state problem

$$w(\underline{x}) I_p + L T_p(\underline{x}) = d_p w(\underline{x}) T_p(\underline{x}) + P_e(\underline{x}), \quad \underline{x} \in V \tag{15a}$$

$$\gamma(\underline{x}) I_p + B T_p(\underline{x}) = d_p \gamma(\underline{x}) T_p(\underline{x}), \quad \underline{x} \in S. \tag{15b}$$

In general $I_p = 0$, except for the case $d(\underline{x}) = \alpha(\underline{x}) = 0$

when

$$I_p = \frac{\int_V P_e(\underline{x})\,\mathrm{d}v}{\int_V w(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s} \tag{15c}$$

and the following additional condition appears

$$\int_V w(\underline{x})T_p(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} T_\phi(\underline{x})\,\mathrm{d}s = 0. \tag{15d}$$

The functions $T_j(\underline{x})$ are the solutions of the following steady-state system

$$w(\underline{x})I_j + LT_j(\underline{x}) + (j+1)w(\underline{x})T_{j+1}(\underline{x}) = P_j(\underline{x}), \quad \underline{x} \in V \tag{16a}$$

$$\gamma(\underline{x})I_j + BT_j(\underline{x}) + (j+1)\gamma(\underline{x})T_{j+1}(\underline{x}) = \phi_j(\underline{x}), \quad \underline{x} \in S \tag{16b}$$

for $j = q, q-1, \ldots, 1, 0$ with

$$T_{q+1}(\underline{x}) = 0. \tag{16c}$$

In general $I_j = 0$, except for the case $d(\underline{x}) = \alpha(\underline{x}) = 0$ when

$$I_j = \frac{\int_V P_j(\underline{x})\,\mathrm{d}v + \int_S \frac{\phi_j(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s}{\int_V w(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s} \tag{16d}$$

and the following additional condition appears

$$\int_V w(\underline{x})T_j(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} T_j(\underline{x})\,\mathrm{d}s = 0. \tag{16e}$$

The transient function $T_t(\underline{x}, t)$ is the solution of the following problem

$$w(\underline{x})\frac{\partial T_t(\underline{x}, t)}{\partial t} + LT_t(\underline{x}, t) = 0, \quad \underline{x} \in V \tag{17a}$$

$$\gamma(\underline{x})\frac{\partial T_t(\underline{x}, t)}{\partial t} + BT_t(\underline{x}, t) = 0, \quad \underline{x} \in S \tag{17b}$$

$$T_t(\underline{x}, 0) = f(\underline{x}) - I_f - T_\phi(\underline{x}) - T_p(\underline{x}) - T_0(\underline{x}), \quad \underline{x} \in V. \tag{17c}$$

In general $I_f = 0$, except for the case $d(x) = \alpha(x) = 0$ when

$$I_f = \frac{\int_V w(\underline{x})f(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})} f_s(\underline{x})\,\mathrm{d}s}{\int_V w(\underline{x})\,\mathrm{d}v + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\,\mathrm{d}s}. \tag{17d}$$

The validity of the foregoing splitting-up process can be readily verified by substituting equation (13) into the problem (1). The additional conditions (14d), (15d) and (16e) result when equation (13), taking into account the additional term defined by equation (11a) is introduced into definition (11b) and the T_{av} term cancelled out.

The solution of the homogeneous problem (17) is obtained from the general solution (10) as

$$\begin{aligned} T_t(\underline{x}, t) = \sum_{i=1}^{\infty} \frac{\psi_i(\underline{x})}{N_i} e^{-\mu_i^2 t} \Big\{ & \int_V w(\underline{x})\psi_i(\underline{x})f(\underline{x})\,\mathrm{d}v \\ & + \int_S \frac{\gamma(\underline{x})}{\beta(\underline{x})}\psi_i(\underline{x})f_s(\underline{x})\,\mathrm{d}s - \frac{1}{\mu_i^2 - d_\phi} \\ & \times \int_S \frac{\phi_e(\underline{x})}{\beta(\underline{x})}\psi_i(\underline{x})\,\mathrm{d}s - \frac{1}{\mu_i^2 - d_p}\int_V \psi_i(\underline{x})P_e(\underline{x})\,\mathrm{d}v \\ & - \sum_{j=0}^{q} \frac{(-1)^j j!}{\mu_i^{2(j+1)}}\Big[\int_S \frac{\phi_j(\underline{x})}{\beta(\underline{x})}\psi_i(\underline{x})\,\mathrm{d}s \\ & + \int_V \psi_i(\underline{x})P_j(\underline{x})\,\mathrm{d}v\Big]\Big\}. \end{aligned} \tag{18}$$

Note that this solution is valid for all cases including $d(\underline{x}) = \alpha(\underline{x}) = 0$, because for this particular case, when taking the integral transform of equation (17c), the contribution of the additional term I_f vanishes.

The advantage of the splitting-up procedure as defined by equation (13) lies in the fact that, the equations defining the functions $T_\phi(x)$, $T_p(x)$ and $T_j(x)$ can be solved by a suitable method such as direct integration, other than the integral transform technique, that provides a rapidly convergent explicit solution for these functions. In addition, the solution (18) converges fast because $T_t(x, t)$ is the solution of the homogeneous problem (17).

ONE-DIMENSIONAL CASE

The one-dimensional problems have numerous important application in practice. As an illustration of the application of the general three-dimensional formal solutions developed above, we now present the one-dimensional problem applicable in the rectangular, cylindrical and spherical coordinates, and the corresponding solutions.

The one-dimensional form of the problem (1) becomes

$$w(x)\frac{\partial T(x,t)}{\partial t} = \frac{\partial}{\partial x}\left\{k(x)\frac{\partial T(x,t)}{\partial x}\right\} - d(x)T(x,t) + P(x,t), \quad \text{in} \quad x_0 < x < x_1, t > 0 \tag{19a}$$

subject to the boundary conditions

$$\alpha_k T(x_k, t) - (-1)^k \beta_k k(x_k)\frac{\partial T(x_k, t)}{\partial x} + \gamma_k \frac{\partial T(x_k, t)}{\partial t} = \phi_k(t), \quad \text{at} \quad x = x_k, \; k = 0, 1, \; t > 0 \tag{19b,c}$$

and the initial conditions

$$T(x, 0) = f(x) \quad \text{in} \quad x_0 < x < x_1, \quad t = 0 \tag{19d}$$

$$T(x_k, 0) = f_k \quad \text{at} \quad x = x_k, \quad t = 0. \tag{19e}$$

Clearly, by appropriate selection of the functional form of the coefficients $w(x)$ and $k(x)$, equation (19a) becomes applicable for the rectangular, cylindrical and spherical coordinates.

The solution of problem (19) is immediately obtainable from the general solution (10) by restricting it to the one-dimensional case. We find

$$T(x,t) = \sum_{i=1}^{\infty} \frac{\psi_i(x)}{N_i} \mathrm{e}^{-\mu_i^2 t} \left\{ \tilde{f}_i + \int_0^t \mathrm{e}^{\mu_i^2 t^*} g_i(t^*) \, \mathrm{d}t^* \right\} \quad (20a)$$

where N_i, $\tilde{f}_i$ and $g_i(t)$ are determined from equations (5), (9) and (8b), respectively as

$$N_i = \int_{x_0}^{x_1} w(x)\psi_i^2(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} \psi_i^2(x_0) + \frac{\gamma_1}{\beta_1} \psi_i^2(x_1) \quad (20b)$$

$$\tilde{f}_i = \int_{x_0}^{x_1} w(x)\psi_i(x) f(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} \psi_i(x_0) f_0 + \frac{\gamma_1}{\beta_1} \psi_i(x_1) f_1 \quad (20c)$$

$$g_i(t) = \int_{x_0}^{x_1} \psi_i(x) P(x,t) \, \mathrm{d}x + \frac{\phi_0(t)}{\beta_0} \psi_i(x_0) + \frac{\phi_1(t)}{\beta_1} \psi_i(x_1). \quad (20d)$$

If $d(x) = \alpha_0 = \alpha_1 = 0$ the solution (20a) includes an additional term corresponding to $\mu_0 = 0$, obtained from equation (11a) as

$$T_{\mathrm{av}}(t) = \left\{ \int_{x_0}^{x_1} w(x) f(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} f_0 + \frac{\gamma_1}{\beta_1} f_1 + \int_0^t \left[\int_{x_0}^{x_1} P(x,t') \, \mathrm{d}x + \frac{\phi_0(t')}{\beta_0} + \frac{\phi_1(t')}{\beta_1} \right] \mathrm{d}t' \right\} \Big/ \left[\int_{x_0}^{x_1} w(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} + \frac{\gamma_1}{\beta_1} \right]. \quad (20e)$$

The eigenfunctions $\psi_i(x)$ and eigenvalues μ_i are determined by the one-dimensional form of the eigenvalue problem (3)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x) \frac{\mathrm{d}\psi(\mu,x)}{x} \right] + [\mu_i^2 w(x) - d(x)]\psi(\mu,x) = 0, \quad \text{in} \quad x_0 < x < x_1 \quad (21a)$$

$$[\alpha_k - \mu_i^2 \gamma_k]\psi_i(x_k) - (-1)^k \beta_k k(x_k) \frac{\mathrm{d}\psi_i(x_k)}{\mathrm{d}x} = 0, \quad k = 1, 2. \quad (21b)$$

When the nonhomogeneous terms of the problem (19) are exponentials and q-order polynomials of time in the form given by equation (12), i.e.

$$P(x,t) = P_e(x)\mathrm{e}^{-d_p t} + \sum_{j=0}^{q} P_j(x) t^j, \quad \text{in} \quad x_0 < x < x_1 \quad (22a)$$

$$\phi_k(t) = \phi_e(x_k)\mathrm{e}^{-d_\phi t} + \sum_{j=0}^{q} \phi_j(x_k) t^j, \quad k = 0, 1 \quad (22b)$$

then the solution of the problem (19) can be split up into the solution of simpler problems as given by equation (13) in the form

$$T(x,t) = T_\phi(x)\mathrm{e}^{-d_\phi t} + T_p(x)\mathrm{e}^{-d_p t} + \sum_{j=0}^{q} T_j(x) t^j + T_t(x,t), \quad \text{in} \quad x_0 < x < x_1. \quad (23)$$

If $\mathrm{d}(x) = \alpha_0 = \alpha_1 = 0$ then in the RHS of equation (23) appears the additional term defined by equation (20e).

The function $T_\phi(x)$ satisfies the one-dimensional form of the problem (14) given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x) \frac{\mathrm{d}T_\phi(x)}{\mathrm{d}x} \right] + [d_\phi w(x) - \mathrm{d}(x)] T_\phi(x) = w(x) I_\phi, \quad \text{in} \quad x_0 < x < x_1 \quad (24a)$$

$$[\alpha_k - d_\phi \gamma_k] T_\phi(x_k) - (-1)^k \beta_k k(x_k) \frac{\mathrm{d}T_\phi(x_k)}{\mathrm{d}x} = \phi_e(x_k) - \gamma_k I_\phi, \quad k = 0, 1 \quad (24b)$$

where $I_\phi = 0$ except for the case $d(x) = \alpha_0 = \alpha_1 = 0$ when equation (14c) becomes

$$I_\phi = \frac{\dfrac{\phi_e(x_0)}{\beta_0} + \dfrac{\phi_e(x_1)}{\beta_1}}{\displaystyle\int_{x_0}^{x_1} w(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} + \frac{\gamma_1}{\beta_1}} \quad (24c)$$

and the additional condition (14d) takes the form

$$\int_{x_0}^{x_1} w(x) T_\phi(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} T_\phi(x_0) + \frac{\gamma_1}{\beta_1} T_\phi(x_1) = 0 \quad (24d)$$

The function $T_p(x)$ satisfies the one-dimensional version of the problem (15) given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x) \frac{\mathrm{d}T_p(x)}{\mathrm{d}x} \right] + [d_p w(x) - d(x)] T_p(x) + P_e(x) = w(x) I_p, \quad \text{in} \quad x_0 < x < x_1 \quad (25a)$$

$$[\alpha_k - d_p \gamma_k] T_p(x_k) - (-1)^k \beta_k k(x_k) \frac{\mathrm{d}T_p(x_k)}{\mathrm{d}x} = -\gamma_k I_p, \quad k = 0, 1 \quad (25b)$$

where $I_p = 0$ except for the case $d(x) = \alpha_0 = \alpha_1 = 0$ when equation (15c) becomes

$$I_p = \frac{\displaystyle\int_{x_0}^{x_1} P_e(x) \, \mathrm{d}x}{\displaystyle\int_{x_0}^{x_1} w(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} + \frac{\gamma_1}{\beta_1}} \quad (25c)$$

and the additional condition (15d) takes the form

$$\int_{x_0}^{x_1} w(x) T_p(x) \, \mathrm{d}x + \frac{\gamma_0}{\beta_0} T_p(x_0) + \frac{\gamma_1}{\beta_1} T_p(x_1) = 0. \quad (25d)$$

The functions $T_j(x)$ satisfy the one-dimemsional form of equations (16) given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[k(x)\frac{\mathrm{d}T_j(x)}{\mathrm{d}x}\right]-d(x)T_j(x)+P_j(x)$$
$$=(j+1)w(x)T_{j+1}(x)+w(x)I_j, \quad \text{in} \quad x_0<x<x_1 \tag{26a}$$

$$\alpha_k T_j(x_k)-(-1)^k\beta_k k(x_k)\frac{\mathrm{d}T_j(x_k)}{\mathrm{d}x}$$
$$=\phi_j(x_k)-\gamma_k[I_j+(j+1)T_{j+1}(x_k)], \quad k=0,1 \tag{26b}$$

for $j=q,q-1,\ldots,1,0$ with

$$T_{q+1}(x)=0 \tag{26c}$$

where $I_j=0$ except for the case $d(x)=\alpha_0=\alpha_1=0$ when equation (16d) becomes

$$I_j=\frac{\displaystyle\int_{x_0}^{x_1}P_j(x)\,\mathrm{d}x+\frac{\phi_j(x_0)}{\beta_0}+\frac{\phi_j(x_1)}{\beta_1}}{\displaystyle\int_{x_0}^{x_1}w(x)\,\mathrm{d}x+\frac{\gamma_0}{\beta_0}+\frac{\gamma_1}{\beta_1}} \tag{26d}$$

and the additional condition (16a) takes the form

$$\int_{x_0}^{x_1}w(x)T_j(x)\,\mathrm{d}x+\frac{\gamma_0}{\beta_0}T_j(x_0)+\frac{\gamma_1}{\beta_1}T_j(x_1)=0. \tag{26e}$$

Finally, the one-dimensional form of the transient solution (18) becomes

$$T_t(x,t)=\sum_{i=1}^{\infty}\frac{\psi_i(x)}{N_i}\mathrm{e}^{-\mu_i^2 t}\left\{\int_{x_0}^{x_1}w(x)\psi_i(x)f(x)\,\mathrm{d}x\right.$$
$$+\sum_{k=0}^{1}\frac{\gamma_k}{\beta_k}\psi_i(x_k)f_k-\frac{1}{\mu_i^2-d_\phi}\sum_{k=0}^{1}\frac{\phi_e(x_k)}{\beta_k}\psi_i(x_k)$$
$$-\frac{1}{\mu_i^2-d_p}\int_{x_0}^{x_1}\psi_i(x)P_e(x)\,\mathrm{d}x-\sum_{j=0}^{q}\frac{(-1)^j j!}{\mu_i^{2(j+1)}}$$
$$\left.\times\left[\sum_{k=0}^{1}\frac{\phi_j(x_k)}{\beta_k}\psi_i(x_k)+\int_{x_0}^{x_1}\psi_i(x)P_j(x)\,\mathrm{d}x\right]\right\}. \tag{27}$$

CONCLUDING REMARKS

The general results presented above for the solution of heat conduction problems involving time derivative in the boundary conditions contain as special cases all the problems belonging to this class that are reported in the literature. For example the problems in chap. 4 of ref. [4] and the solutions developed in the ref. [5] are merely the special cases. To develop analytic solutions to a one-dimensional problem, the correspondence between the general one-dimensional problem (19) and the specific problem is established by comparing the differential equation, the boundary and initial conditions. By utilizing the results of this correspondence regarding the values of various coefficients, the solution to the specific problem is obtained by appropriate simplification of the general solution (20) or its split up from (23). The corresponding eigenvalue problem is obtained by the proper simplification of the eigenvalue problem (21). Practical aspects of such applications are discussed in detail in ref. [1], where only the case $\gamma(x)=0$ is considered.

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REFERENCES

1. M. D. Mikhailov and M. N. Özişik, *Unified Analysis and Solutions of Heat and Mass Diffusion.* John Wiley, New York, 1984.
2. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids.* Oxford University Press, London, 1959.
3. J. Crank, *The Mathematics of Diffusion*, 2nd ed. Clarendon Press, London, 1975.
4. M. D. Mikhailov, *Nonstationary Temperature Fields in Skin.* Energia, Moscow (in Russian), 1967.
5. J. V. Beck, Green's function solution for transient heat conduction problems, *Int. J. Heat Mass Transfer* **27**, 1235–1244, 1984.

SOLUTIONS UNIFIEES DE LA DIFFUSION THERMIQUE DANS UNE REGION FINIE COMPRENANT UN FILM DE CAPACITE THERMIQUE FINIE

Résumé—La technique de transformée intégrale finie est développée et utilisée pour obtenir une solution exacte de la diffusion thermique variable dans une région finie arbitraire ayant un film en surface de capacité thermique finie. On décrit une procédure de séparation qui accélère la convergence des solutions en série utilisées ici. Les solutions d'un grand nombre de situations spécifiques, intéressantes en pratique sont obtenues comme des cas spéciaux des résultats généraux présentés dans ce texte.

VERALLGEMEINERTE LÖSUNGEN DER WÄRMEDIFFUSION IN EINEM BEGRENZTEN GEBIET MIT EINEM OBERFLÄCHENFILM VON ENDLICHER WÄRMEKAPAZITÄT

Zusammenfassung—Die Transformationstechnik der endlichen Integrale wurde weiterentwickelt und angewandt, um eine exakte Lösung für die instationäre Wärmediffusion in einem beliebigen begrenzten Gebiet mit einem Oberflächenfilm von endlicher Wärmekapazität zu erhalten. Ein Abspaltungsverfahren, welches die Konvergenz der Lösungen der hier entwickelten Reihen beschleunigt, wird beschrieben. Lösungen für eine riesige Anzahl von speziellen Situationen des praktischen Interesses können leicht als Spezialfälle aus den allgemeinen Lösungen, die in dieser Arbeit vorgestellt werden, ermittelt werden.

УНИФИЦИРОВАННЫЕ РЕШЕНИЯ ДЛЯ ДИФФУЗИИ ТЕПЛА В ОГРАНИЧЕННОЙ ОБЛАСТИ, ВКЛЮЧАЮЩЕЙ ПОВЕРХНОСТНУЮ ПЛЕНКУ КОНЕЧНОЙ ТЕПЛОЕМКОСТИ

Аннотация—Развитая в работе методика конечных интегральных преобразований применяется для получения точного решения нестационарной термодиффузии в произвольной финитной области с поверхностной пленкой конечной теплоемкости. Описывается разработанная методика, ускоряющая сходимость рядных решений. Для большого числа типичных ситуаций, имеющих практический интерес, решения могут быть легко получены как частные случаи общих результатов, представленных в работе.